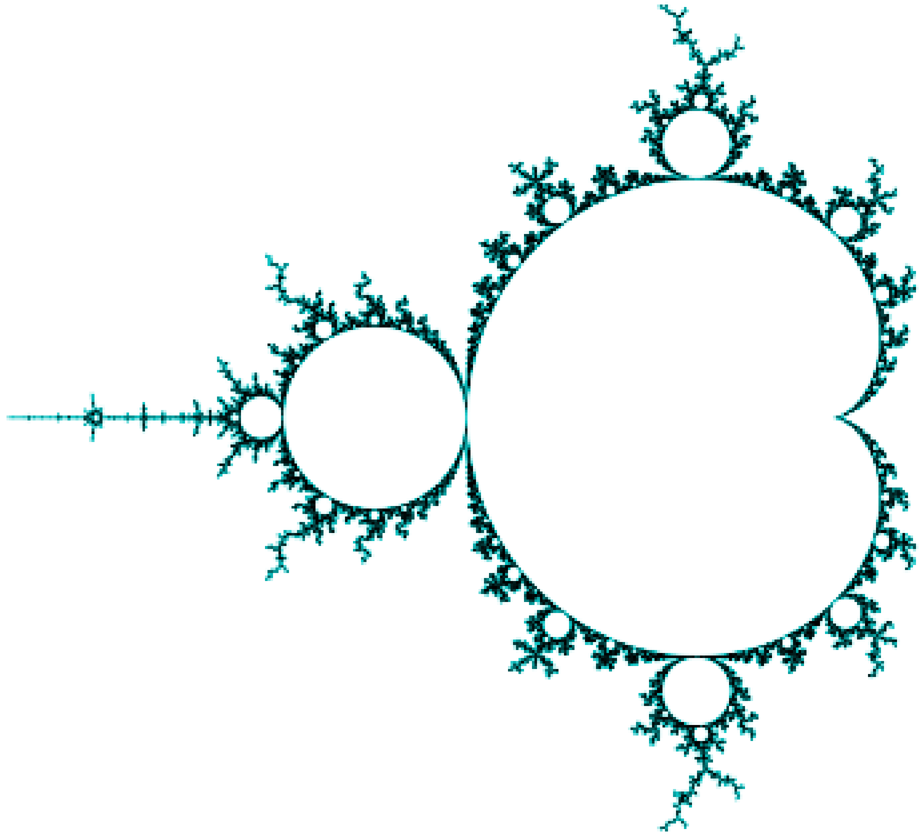
# Further Mathematics Induction Booklet



## Summer reading:

You are encouraged to read the following two books over the summer

- The Story of Mathematics by Anne Rooney (ISBN 978- 1- 84193 -940-7)
- The Calculus Wars by Jason Bardi (ISBN 978-1-84344-036-9)

## General Introduction:

The background to the following exercises is GCSE and it's essential that you are confident with these basic concepts in order to be prepared for the new world of further mathematics and of course the Further Maths induction test in September.

## Instructions:

Please ensure you complete all questions and do any further practice if you're not 100% confident with any parts of the topics. There are explanations on each topic that should act as guidelines on how to approach the questions and correctly set out your work, which is essential for the ground of your studies in the following year.

If you struggle with any part of this, here are the recommended websites where you can get extra support:

- My Maths
- Dr. Frost Maths
- Mr. Barton Maths
- Corbett Maths

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# Questions

## Trigonometry:

### Trigonometric Equations

You can of course get one solution to an equation such as $\sin x = -0.5$ from your calculator. But what about others?

***Example 1*** Solve the equation $\sin x° = -0.5$ for $0 \le x < 360$.

***Solution*** The calculator gives $\sin^{-1}(0.5) = -30$.

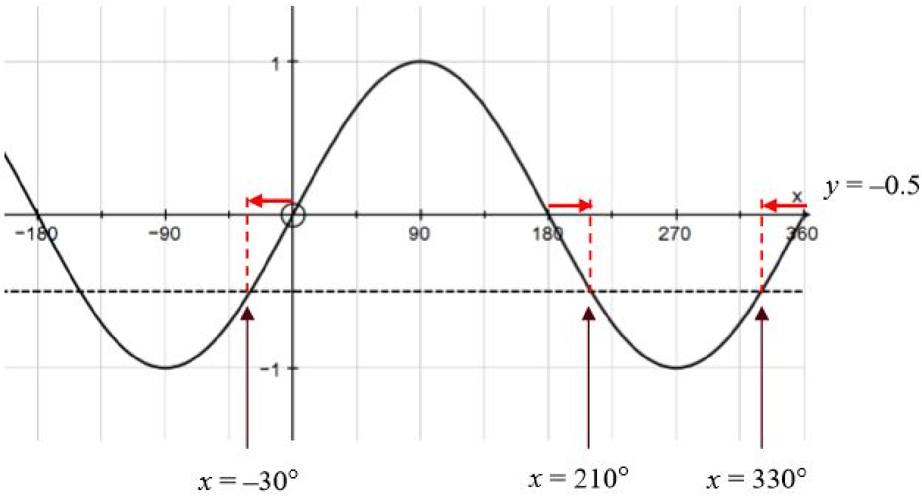
This is usually called the *principal value* of the function $\sin^{-1}$.

To get a second solution you can either use a graph or a standard rule.

***Method 1:*** Use the graph of $y = \sin x$

By drawing the line $y = -0.5$ on the same set of axes as the graph of the sine curve, points of intersection can be identified in the range

$0 \le x < 360$.




***Method 2:*** Use an algebraic rule.

To find the second solution you use $\sin(180 - x)° = \sin x°$

$\tan(180 + x)° = \tan x°$

$\cos(360 - x)° = \cos x°$.

Any further solutions are obtained by adding or subtracting 360 from the principal value or the second solution.

In this example the principal solution is $-30°$.

Therefore, as this equation involves sine, the second solution is:

$$180 - (-30)° = 210°$$

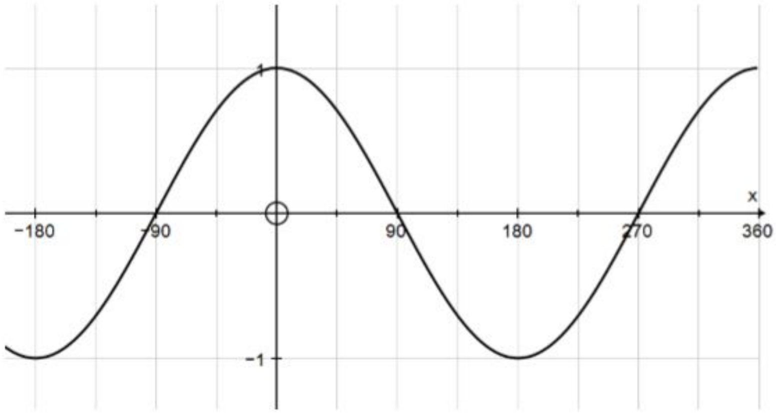
$-30°$ is not in the required range, so add 360 to get:

$$360 + (-30) = 330°.$$

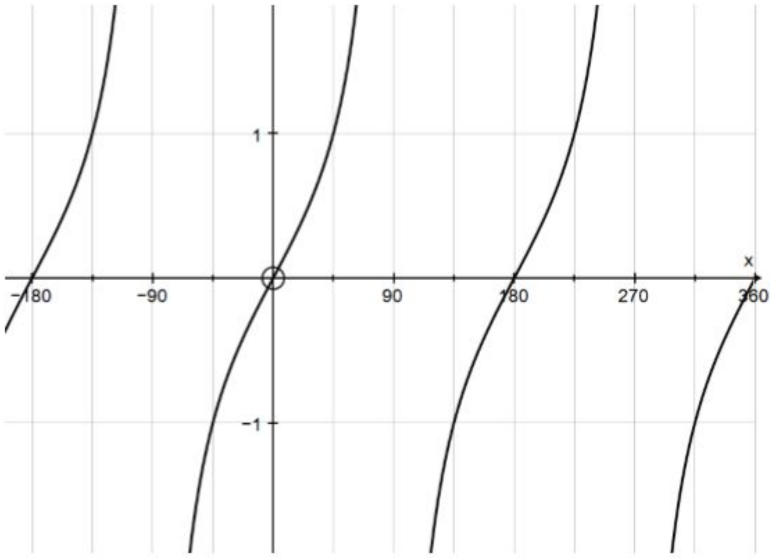
Hence the required solutions are $210°$ or $330°$.

You should decide which method you prefer. The corresponding graphs for $\cos x$ and $\tan x$ are shown below.

$y = \cos x$




$y = \tan x$




To solve equations of the form $y = \sin(kx)$, you will expect to get $2k$ solutions in any interval of 360°. You can think of compressing the graphs, or of using a wider initial range.

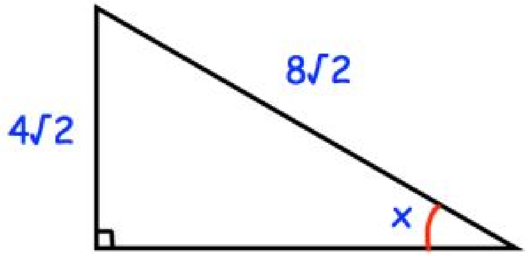
Here are the exact trig values that you should have learnt for GCSE:

| | 0° | 30° | 45° | 60° | 90° |
|---|---|---|---|---|---|
| sin | 0 | $\frac{1}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{\sqrt{3}}{2}$ | 1 |
| cos | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{1}{\sqrt{2}}$ | $\frac{1}{2}$ | 0 |
| tan | 0 | $\frac{1}{\sqrt{3}}$ | 1 | $\sqrt{3}$ | — |

Practice Section A:

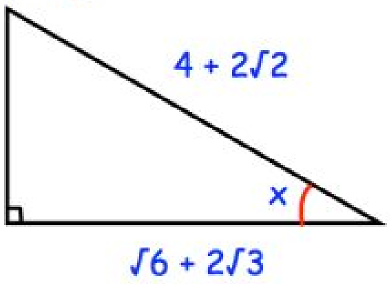
The following questions are about exact trig values, please do them without a calculator!

a) Below is a right angled triangle.




Show that angle $x = 30°$
Include all your working.

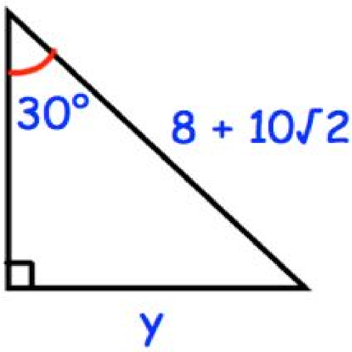
b) Below is a right angled triangle.




Show that angle $x = 30°$
Include all your working.

c) Find the exact value of $\mathrm{Sin}(45°) + \mathrm{Cos}(30°)$

d) Shown below is a right angled triangle.




Find the exact length of the side labelled y.

# Algebraic fractions:

If you know that $x^2$ divided by $x = x$, this section will be quite simple for you. Algebraic fractions are the same as normal fractions, just with letters.

## How to simplify algebraic fractions:

When simplifying fractions, the aim is to find a common factor in the top and the bottom and then cancel them out, e.g. we get that 6/8 = 3/4 by observing 6 and 8 have a common factor of 2, which we can then cancel from the top and bottom (in other words, we divide both numbers by it). Here, the aim is the same, but factors look a little different.

Note: It may be the case that you have to do some expanding first before you can factorise. Bear this in mind when you're having a go at the questions below.

Example: Simplify fully the fraction below

$$\frac{a^2 + a - 6}{ab + 3b}.$$

First, we will consider factorising the numerator. We see that is a quadratic, so observing that the numbers 3 and -2 multiply to make -6 and add to make 1, we get

$a^2 + a - 6 = (a + 3)(a - 2)$

Now, for the denominator:

ab + 3b = b(a + 3)

This might seem insignificant, but what we now have is a factor of (a+3) in both the numerator and the denominator, which means we can cancel it. This looks like:

$$\frac{a^2 + a - 6}{ab + 3b} = \frac{(a + 3)(a - 2)}{b(a + 3)} = \frac{a - 2}{b}$$

There is nothing else that the top and bottom have in common, so that must mean that we're done.

## Adding and subtracting algebraic fractions:

When adding or subtracting algebraic fractions, the aim is still to find a common denominator. If you can, it might make your life a little easier to use the lowest common multiple of both the denominators in question as your common denominator. However, it isn't always easy to know what this is so if you're unsure, you can always safely use the product of both denominators instead (and to be honest, in algebraic fraction this often is the lowest common multiple anyway).

**Example:**

$$\frac{m}{m-6}+\frac{m}{7}$$

Our choice of common denominator will be the product of both denominators, which is

7×(m−6)=7(m−6)

We could expand the brackets, but it's better to leave it factorised in case we can cancel anything later. So, our left-hand fraction is being multiplied by 7 on top and bottom, whilst our right-hand fraction is being multiplied by (m - 6)(m−6). So, we get

$$\frac{m}{m-6}+\frac{m}{7}=\frac{7m}{7(m-6)}+\frac{m(m-6)}{7(m-6)}$$
$$=\frac{7m+m(m-6)}{7(m-6)}$$

We have written it as one fraction, now we must simplify it fully. First, we should expand the brackets and collect terms on the top, which we can then factorize to see if there are any common factors we can cancel.

Denominator:

$$7m + m(m - 6) = 7m + m^2 - 6m = m^2 + m = m(m + 1)$$

We cannot factorize this denominator further, so our fraction becomes

$$\frac{m(m+1)}{7(m-6)}$$

## Practice Section A:

1. Express each of the following as a **single fraction** in its **simplest form**:

a) $\dfrac{a^2}{ab}$ b) $\dfrac{l}{m}\times\dfrac{m}{n}$ c) $\dfrac{3x}{y}\div\dfrac{x}{z}$ d) $\dfrac{6}{a}\div\dfrac{3}{a}$

e) $\dfrac{x(y-1)}{y(x+3)}\times\dfrac{y}{x}$ f) $\dfrac{x}{y}\div\dfrac{x(x+1)}{y(y+2)}$ g) $\dfrac{a-3}{5}\times\dfrac{10}{(a-3)(a+1)}$ h) $\dfrac{y-1}{4}\div\dfrac{(y-1)(y+3)}{8}$ **[10]**

2. Express each of the following as a **single fraction** in its **simplest form**:

a) $\dfrac{(5x+5)(x-2)}{(x+1)(2x-4)}$ b) $\dfrac{(y^2+2y-3)(y-4)}{(y+3)(y^2-3y-4)}$

c) $\dfrac{a^2+6a+5}{a^2-a-6}\times\dfrac{a+2}{a+5}$ d) $\dfrac{x^2-2x-8}{x+2}\div\dfrac{x^2+x-20}{x-3}$ **[9]**

3. Write each of the following as a **single fraction**:

a) $\dfrac{1}{4x}+\dfrac{1}{y}$ b) $\dfrac{2}{a}-\dfrac{3}{a+2}$ c) $\dfrac{5}{x-1}+\dfrac{3}{2x-2}$ **[8]**

## Practice Section B:

1. Express each of the following as a **single fraction** in its **simplest form**:

a) $\dfrac{ab}{ac}$ b) $\dfrac{x}{y}\times\dfrac{y}{z}$ c) $\dfrac{2m}{n}\div\dfrac{m}{p}$ d) $\dfrac{4}{x}\div\dfrac{5}{x}$

e) $\dfrac{x(x+1)}{y(y+2)}\times\dfrac{y}{x}$ f) $\dfrac{a}{b}\div\dfrac{a(a+4)}{b(b-2)}$ g) $\dfrac{x+3}{2}\times\dfrac{4}{(x+4)(x+3)}$ h) $\dfrac{y+2}{3}\div\dfrac{(y+2)(y+5)}{9}$ **[10]**

2. Express each of the following as a **single fraction** in its **simplest form**:

a) $\dfrac{(3x-9)(x+1)}{(x-3)(2x+2)}$ b) $\dfrac{(y^2+4y+3)(y+4)}{(y+1)(y^2+6y+8)}$

c) $\dfrac{a^2+5a+6}{a^2+3a-4}\times\dfrac{a+4}{a+2}$ d) $\dfrac{x^2+7x+10}{x+5}\div\dfrac{x^2+3x+2}{x+3}$ **[9]**

3. Write each of the following as a **single fraction**:

a) $\dfrac{1}{x}+\dfrac{x}{2y}$ b) $\dfrac{3}{a}-\dfrac{2}{a+1}$ c) $\dfrac{2}{x+3}+\dfrac{5}{2x+6}$ **[8]**

Practice Section C:

1. Express each of the following as a **single fraction** in its **simplest form**:

a) $\dfrac{3x^2}{xy^2}$ b) $\dfrac{x}{y(y-4)} \times \dfrac{y(y+2)}{x(x-1)}$ c) $\dfrac{x(x-3)}{y(y-2)} \div \dfrac{x}{y(y+6)}$ **[5]**

2. Express each of the following as a **single fraction** in its **simplest form**:

a) $\dfrac{2}{3x} + \dfrac{x+1}{6x}$ a) $\dfrac{3}{a^2-2a-15} - \dfrac{1}{a^2-4a-5}$ b) $\dfrac{5b-2}{b^2-4b-12} + \dfrac{4}{b+2}$ **[10]**

Practice Section D:

1. Write each of the following as a **single fraction** in its **simplest form**:

a) $\dfrac{a}{b} \div \dfrac{c}{d}$ b) $\dfrac{ab}{3bc}$ c) $\dfrac{8x}{2x^2}$ d) $\dfrac{m(m-2)}{n(n+3)} \times \dfrac{n}{m}$ e) $\dfrac{x(x-4)}{y(y+1)} \div \dfrac{2x}{y}$

f) $\dfrac{x+2}{y} \times \dfrac{3}{(x-1)(x+2)}$ g) $\dfrac{x}{y-5} \div \dfrac{2z}{(y-5)(y+3)}$ h) $\dfrac{a(a-1)}{b(b+6)} \times \dfrac{b}{a(a+6)}$

*Hint for part h): cancel common terms, and do not expand.* **[11]**

2. Express each of the following as a **single fraction** in its **simplest form**:

a) $\dfrac{(x^2-4x+3)(y+2)}{(x-3)(y^2-y-6)}$ b) $\dfrac{(a^2-5a+4)(a+2)}{(a^2+5a+6)(a-4)}$

c) $\dfrac{x^2-4}{3} \times \dfrac{3}{x^2+3x+2}$ d) $\dfrac{x^2+3x-4}{x^2+4x+4} \times \dfrac{x+2}{x^2-4x+3}$ **[10]**

3. Write each of the following as a **single fraction**:

a) $\dfrac{4}{y+1} + \dfrac{3}{y-2}$ b) $\dfrac{2}{x+2} - \dfrac{1}{x-1}$ c) $\dfrac{1}{m^2+m-6} - \dfrac{1}{m^2-3m+2}$ **[10]**

## Proofs

The idea of a proof is to make a universal statement – for example, you don't just want to say that the angles in *some* triangles add up to 180, you want to say that the angles in **all** triangles add up to 180.

The general structure of a proof is to begin with one statement, take a series of logical and mathematical steps, and end up at the desired conclusion. Of course, not everything we want can be proved true. In fact, all it takes to prove a statement false is to find a **counterexample** – a particular example for which the universal statement you're trying to make doesn't hold true.

**Example 1 (counterexample):** Hernan claims "if you square a number and add 1, the result is a prime number". Find a counterexample to prove her statement wrong.

$1^2$ +1=1+1=2, which is prime

$2^2$ +1=4+1=5, which is prime

$3^2$ +1=9+1=10, which is not prime

This is a counterexample for her statement, so we have proved it to be false.

**Example 2 (divisibility):** Prove that $(n + 2)^2 - (n - 2)^2$ is divisible by 8 for any positive whole number n.

To do this, we need to show that $(n + 2)^2 - (n - 2)^2$ can be written in some way that is clearly divisible by 8. To find a way to write an expression like this differently, we can try expanding it. So, the first bracket expands to:

$$(n+2)^2 = n^2 + 2n + 2n + 4 = n^2 + 4n + 4.$$

Then, the second bracket expands to

$$(n-2)^2 = n^2 - 2n - 2n + 4 = n^2 - 4n + 4.$$

The expression in the question has the second bracket being subtracted from the first one. So, we will do this subtraction with the expansion of the two brackets:

$$(n+2)^2 - (n-2)^2 = (n^2 + 4n + 4) - (n^2 - 4n + 4)$$

We can see that the n^2*n*2 terms will cancel, as will the 4s, so all we're left with is:

$$(n^2 + 4n + 4) - (n^2 - 4n + 4) = 4n - (-4n) = 8n$$

So, the whole expression simplifies to 8n. Now, if n is a whole number, then 8n must be divisible by 8 (if we divide it by 8, we get the answer n). Since 8n is equivalent to the expression we started with, it must be the case that $(n + 2)^2 - (n - 2)^2$ is divisible by 8 for any positive whole number n – so the statement is now universal. Thus, we have completed the proof.

**Example 3 (odd numbers):** Prove that the square of an odd number is also odd.

Okay, so if we're going to make a statement about odd numbers and we want to use algebra to prove the statement, we need some way to express an odd number with algebra.

To do this, let's consider even numbers: an even number is a multiple of 2, by definition, so if n is any whole number, then 2n must be an even number. Indeed, any even number can be expressed by 2n if you pick the right n. Furthermore, any odd number is always the next one along from an even number. So, we can express an odd number as $2n+1$.

So, now that we have our algebraic expression of an odd number, we can really get going. The question asks about the square of an odd number, so let's square our expression for a general odd number:

$$(2n+1)^2 = 4n^2 + 2n + 2n + 1 = 4n^2 + 4n + 1.$$

The question is now: how do we know this is odd? Well, we can't take a factor of 2 out of the whole thing, but we can take a factor of 2 out of the first two terms. Doing so, we get:

$$4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$$

Now, $2n^2 + 2n$ might seem like a reasonably complicated expression but importantly, it must be a whole number. This is because n is a whole number, and if you square a whole number/multiply it by other whole numbers, the result must still be a whole number.

$$2(2n^2 + 2n) + 1 \quad \text{is just} \quad 2 \times (\text{some whole number}) + 1.$$

And since 2 x (some whole number) is how we defined an even number, it follows that 2 x (some whole number) + 1 is how we define an odd number.

Therefore, this expression – and thus, the square of <u>any</u> odd number – must be odd. This is our universal statement, and we have completed the proof.

## Proof Practice Section A:

1. Prove algebraically that

   $(2n + 1)^2 - (2n + 1)$ is an even number

   for all positive integer values of $n$.

2. $c$ is a positive integer.

   Prove that $\frac{6c^3 + 30c}{3c^2 + 15}$ is an even number.

3. a) Prove that the sum of four consecutive whole numbers is always even.

4. Here are the first five terms of an arithmetic sequence.

   7 13 19 25 31

   Prove that the difference between the squares of any two terms of the sequence is always a multiple of 24

## Proof Practice Section B:

**2)** If n is a positive integer, which of the following numbers is always even? [1]

**A.** $11n - 3$ **B.** $n + 4$ **C.** $10n + 4$ **D.** $4n^2 + 1$

**3)** If n is a positive integer, which of the following shows two consecutive square numbers? [1]

**A.** $n^2$ and $n^2 - 2$ **B.** $n^2$ and $(n + 1)^2$

**C.** $n^2$ and $(n + 6)^2$ **D.** $n^2$ and $n^2 + 6$

**4)** If n is a positive integer, then $4n + 8$ is a multiple of:- [1]

**A.** 6 **B.** 3 **C.** 7 **D.** 4

**5)** Prove, using algebra, that the sum of two consecutive whole numbers is always an odd number. [1]

**6)** Prove that $(5n + 4)^2 - (5n - 4)^2$ is a multiple of 4, for all positive integer values of n. [1]

**7)** Prove that $(5n + 2)^2 - (5n - 2)^2$ is a multiple of 8, for all positive integer values of n. [1]

**8)** Prove algebraically that the difference between the squares of any two consecutive integers is equal to the sum of these two integers. [1]

# Matrices - Introduction:

Matrices, and in particular matrix algebra is used in many branches of mathematics. It used extensively in 3D computer graphics, they are also used to manage large amounts of data, or in practical applications such as aviation management control.

Make sure that you set your work out in a clear and logical manner. It should be clearly labeled so that you can find each exercise easily in September.

**Definition:** A matrix (plural: matrices) is an array of numbers, which is displayed in a rectangular table. The number of rows and columns a matrix has is called the dimension of the matrix. If it has ***n*** rows and ***m*** columns it is said to be an ***n*** by ***m*** matrix or ***n*** x ***m*** matrix. Each number in the matrix is called an element of the matrix.

**Examples**

$\begin{pmatrix} 4 & 1 & 0 \end{pmatrix}$ is a 1 by 3 matrix

$\begin{pmatrix} -1 & 4 & 6 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$ is a 3 by 3 matrix

$\begin{pmatrix} 1 & 0 \\ 9 & 6 \\ -5 & -2 \end{pmatrix}$ is a 3 by 2 matrix

Matrices can be added or subtracted by adding or subtracting the corresponding elements. Matrices can only be added or subtracted if they are the same dimension.

**Examples**

$$\begin{pmatrix} 3 & 2 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 3+2 & 2+(-3) \\ 0+4 & -2+1 \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 1 \\ 2 & 0 \\ 0 & 9 \end{pmatrix} - \begin{pmatrix} 4 & -3 \\ 1 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 1 & -2 \\ 1 & 9 \end{pmatrix}$$

**Matrices Practice Section A:**

Complete the following questions

**1** Describe the dimensions of these matrices.

**a** $\begin{pmatrix} 1 & 0 \\ -1 & 3 \end{pmatrix}$

**b** $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

**c** $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & -1 \end{pmatrix}$

**d** $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$

**e** $\begin{pmatrix} 3 & -1 \end{pmatrix}$

**f** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

**2** For the matrices

$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 4 & 1 \\ -1 & -2 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$,

find

**a** $\mathbf{A} + \mathbf{C}$

**b** $\mathbf{B} - \mathbf{A}$

**c** $\mathbf{A} + \mathbf{B} - \mathbf{C}$.

**3** For the matrices

$\mathbf{A} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}$,

$\mathbf{D} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}$, $\mathbf{E} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\mathbf{F} = \begin{pmatrix} 2 & 1 & 3 \end{pmatrix}$,

find where possible:

**a** $\mathbf{A} + \mathbf{B}$

**b** $\mathbf{A} - \mathbf{E}$

**c** $\mathbf{F} - \mathbf{D} + \mathbf{C}$

**d** $\mathbf{B} + \mathbf{C}$

**e** $\mathbf{F} - (\mathbf{D} + \mathbf{C})$

**f** $\mathbf{A} - \mathbf{F}$

**g** $\mathbf{C} - (\mathbf{F} - \mathbf{D})$.

**4** Given that $\begin{pmatrix} a & 2 \\ -1 & b \end{pmatrix} - \begin{pmatrix} 1 & c \\ d & -2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$, find the values of the constants $a$, $b$, $c$ and $d$.

**5** Given that $\begin{pmatrix} 1 & 2 & 0 \\ a & b & c \end{pmatrix} + \begin{pmatrix} a & b & c \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} c & 5 & c \\ c & c & c \end{pmatrix}$, find the values of $a$, $b$ and $c$.

**6** Given that $\begin{pmatrix} 5 & 3 \\ 0 & -1 \\ 2 & 1 \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ 2 & 0 \\ 1 & 4 \end{pmatrix}$, find the values of $a$, $b$, $c$, $d$, $e$ and $f$.

## Scalar Multiplication.

A scalar is a number. Scalar multiplication is when we multiply each element within the matrix by the same number.

**Example**
$$4\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}4\\0\\8\end{pmatrix}$$

$$\begin{pmatrix}2&1\\1&3\end{pmatrix}+3\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}5&1\\1&6\end{pmatrix}$$

**Matrices Practice Section B:**

Complete the following questions:-

**1** For the matrices $\mathbf{A}=\begin{pmatrix}2&0\\4&-6\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1\\-1\end{pmatrix}$, find

**a** $3\mathbf{A}$ **b** $\frac{1}{2}\mathbf{A}$ **c** $2\mathbf{B}$.

**2** Find the value of $k$ and the value of $x$ so that $\begin{pmatrix}0&1\\2&0\end{pmatrix}+k\begin{pmatrix}0&2\\-1&0\end{pmatrix}=\begin{pmatrix}0&7\\x&0\end{pmatrix}$.

**3** Find the values of $a$, $b$, $c$ and $d$ so that $2\begin{pmatrix}a&0\\1&b\end{pmatrix}-3\begin{pmatrix}1&c\\d&-1\end{pmatrix}=\begin{pmatrix}3&3\\-4&-4\end{pmatrix}$.

**4** Find the values of $a$, $b$, $c$ and $d$ so that $\begin{pmatrix}5&a\\b&0\end{pmatrix}-2\begin{pmatrix}c&2\\1&-1\end{pmatrix}=\begin{pmatrix}9&1\\3&d\end{pmatrix}$.

**5** Find the value of $k$ so that $\begin{pmatrix}-3\\k\end{pmatrix}+k\begin{pmatrix}2k\\2k\end{pmatrix}=\begin{pmatrix}k\\6\end{pmatrix}$.

## Matrix Multiplication

Two matrices can only be multiplied together if the number of columns in the first matrix is equal to the number of rows in the second. The resulting matrix then has the number of rows from the first matrix and the number of columns from the second.

How matrices are multiplied together is not intuitive. The method is given in general terms for 2 by 2 matrices and examples given for matrices of different dimensions.

Work along the first row of the first matrix while going down the first column of the second matrix, then do the first row & the second column, second row & first column, second row & second column.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

**Examples**

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2+0 & 3+0 \\ 4+1 & 6+2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 5 & 8 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1+0+3 & 0+4+3 \end{pmatrix} = \begin{pmatrix} 4 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 3 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 8 \\ 2 & 8 \end{pmatrix}$$

**Important Example**

If A = $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ find AB and also BA

## Non Commutativity of Matrix Multiplication

You should have found in the example above that the answers AB and BA are not the same.

We say that matrix multiplication is not commutative. (Unlike normal numbers where the order in which we multiply them does not matter, 3 x 2 = 2 x 3).

Therefore, the **order** in which we multiply matrices is important and does matter.

## **Matrices Practice Section C:**

Complete the following questions

**1** Given the dimensions of the following matrices:

| Matrix | **A** | **B** | **C** | **D** | **E** |
|---|---|---|---|---|---|
| Dimension | $2 \times 2$ | $1 \times 2$ | $1 \times 3$ | $3 \times 2$ | $2 \times 3$ |

Give the dimensions of these matrix products.

**a** **BA** **b** **DE** **c** **CD**

**d** **ED** **e** **AE** **f** **DA**

**2** Find these products.

**a** $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 5 \\ -1 & -2 \end{pmatrix}$

**3** The matrix $\mathbf{A} = \begin{pmatrix} -1 & -2 \\ 0 & 3 \end{pmatrix}$ and the matrix $\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

Find

**a** **AB** **b** $\mathbf{A}^2$

$\mathbf{A}^2$ means $\mathbf{A} \times \mathbf{A}$

**4** The matrices **A**, **B** and **C** are given by

$$\mathbf{A} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} -3 & -2 \end{pmatrix}.$$

Determine whether or not the following products are possible and find the products of those that are.

**a** **AB** **b** **AC** **c** **BC**

**d** **BA** **e** **CA** **f** **CB**

**5** Find in terms of $a$ $\begin{pmatrix} 2 & a \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 2 \end{pmatrix}$.

**6** Find in terms of $x$ $\begin{pmatrix} 3 & 2 \\ -1 & x \end{pmatrix}\begin{pmatrix} x & -2 \\ 1 & 3 \end{pmatrix}$.

## The Identity Matrix

I = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, is called the identity matrix. So also is any other square matrix which has ones down the leading diagonal and zeros elsewhere. Define the "leading diagonal"

If A = $\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$.

Find the matrices AI, IA, BI, IB IAB, AIB, ABI.
**Then write down the effect of the identity matrix when multiplying with other matrices.**

## The inverse of a 2 by 2 matrix

We wish to find a matrix so that when we multiply it by A we obtain the identity matrix, I. This matrix we wish to find is called the inverse of A.

If A= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then its inverse is given by $A^{-1}$ = $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

ad – bc is called the determinant of the matrix A. We write it as **detA**

If det A = 0 then the inverse of A cannot be found and the matrix is said to be a **singular matrix.**

<u>Example:</u>
Show using A and $A^{-1}$ above that $A^{-1}$ A = I.

$AA^{-1} = \frac{1}{ad-bc}\,(a\ b\ c\ d\,)(d\ -b\ -c\ a\,)$
$= \frac{1}{ad-bc}\,(ad-bc\ \ -ab+ba\ cd-dc\ \ -bc+ad\,)$
$= \frac{1}{ad-bc}\,(ad-bc\ 0\ 0\ ad-bc\,)$
$= \frac{1}{ad-bc}\,(1\ 0\ 0\ 1\,)$

Similarly, it can be shown that A $A^{-1}$ = I.

## Using the inverse of a matrix

Unlike with numbers, the inverse of multiplication by a matrix is not division by a matrix; it is multiplication by the inverse of the matrix.

<u>Example:</u>
Given that **A** = $\begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$ and **AB** = $\begin{pmatrix} 3 & 6 \\ 1 & 22 \end{pmatrix}$, find **B**.

To find **B**, we must pre-multiply (multiply on the left) both sides by the inverse of **A**, $\mathbf{A}^{-1}$.
$\mathbf{A}^{-1}$ = $\frac{1}{10}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}$

**AB** = $\begin{pmatrix} 3 & 6 \\ 1 & 22 \end{pmatrix}$
And so $\mathbf{A}^{-1}\mathbf{AB}$ = $\frac{1}{10}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 1 & 22 \end{pmatrix}$
Hence **B** = $\frac{1}{10}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 1 & 22 \end{pmatrix}$

So, **B** = $\frac{1}{10}\begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 3 & 6 \\ 1 & 22 \end{pmatrix}$
**B** = $\frac{1}{10}\begin{pmatrix} 10 & 40 \\ -10 & 20 \end{pmatrix}$
**B** = $\begin{pmatrix} 1 & 4 \\ -1 & 2 \end{pmatrix}$

Note, because multiplication of matrices is non-commutative, we must pre-multiply both sides in this case rather than post-multiply (multiply on the right).

## Matrices Practice Section D:

**1** Determine which of these matrices are singular and which are non-singular. For those that are non-singular find the inverse matrix.

**a** $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 3 \\ -1 & -1 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & 5 \\ 0 & 0 \end{pmatrix}$

**d** $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ **e** $\begin{pmatrix} 6 & 3 \\ 4 & 2 \end{pmatrix}$ **f** $\begin{pmatrix} 4 & 3 \\ 6 & 2 \end{pmatrix}$

**2** Find the value of $a$ for which these matrices are singular.

**a** $\begin{pmatrix} a & 1+a \\ 3 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 1+a & 3-a \\ a+2 & 1-a \end{pmatrix}$ **c** $\begin{pmatrix} 2+a & 1-a \\ 1-a & a \end{pmatrix}$

**6** **a** Given that $\mathbf{BAC} = \mathbf{B}$, where $\mathbf{B}$ is a non-singular matrix, find an expression for $\mathbf{A}$.

**b** When $\mathbf{C} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$, find $\mathbf{A}$.

**7** The matrix $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$ and $\mathbf{AB} = \begin{pmatrix} 4 & 7 & -8 \\ -8 & -13 & 18 \end{pmatrix}$. Find the matrix $\mathbf{B}$.

**8** The matrix $\mathbf{B} = \begin{pmatrix} 5 & -4 \\ 2 & 1 \end{pmatrix}$ and $\mathbf{AB} = \begin{pmatrix} 11 & -1 \\ -8 & 9 \\ -2 & -1 \end{pmatrix}$. Find the matrix $\mathbf{A}$.

**9** The matrix $\mathbf{A} = \begin{pmatrix} 3a & b \\ 4a & 2b \end{pmatrix}$, where $a$ and $b$ are non-zero constants.

**a** Find $\mathbf{A}^{-1}$.

The matrix $\mathbf{B} = \begin{pmatrix} -a & b \\ 3a & 2b \end{pmatrix}$ and the matrix $\mathbf{X}$ is given by $\mathbf{B} = \mathbf{XA}$.

**b** Find $\mathbf{X}$.

# Complex Numbers

The quadratic equation $ax^2 + bx + c = 0$ has solutions given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If the expression under the square root is negative, there are no real solutions.

**Links** For the equation $ax^2 + bx + c = 0$, the **discriminant** is $b^2 - 4ac$.
- If $b^2 - 4ac > 0$, there are two distinct real roots.
- If $b^2 - 4ac = 0$, there are two equal real roots.
- If $b^2 - 4ac < 0$, there are no real roots.

← **Pure Year 1, Section 2.5**

You can find solutions to the equation in all cases by extending the number system to include $\sqrt{-1}$. Since there is no real number that squares to produce $-1$, the number $\sqrt{-1}$ is called an **imaginary number**, and is represented using the letter **i**. Sums of real and imaginary numbers, for example $3 + 2\text{i}$, are known as **complex numbers**.

- **$\text{i} = \sqrt{-1}$**
- **An imaginary number is a number of the form $b\text{i}$, where $b \in \mathbb{R}$.**
- **A complex number is written in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$.**

**Notation** The set of all complex numbers is written as $\mathbb{C}$.
For the complex number $z = a + b\text{i}$:
- $\text{Re}(z) = a$ is the real part
- $\text{Im}(z) = b$ is the imaginary part

**Example 1**

Write each of the following in terms of i.

**a** $\sqrt{-36}$ **b** $\sqrt{-28}$

**a** $\sqrt{-36} = \sqrt{36 \times (-1)} = \sqrt{36}\sqrt{-1} = 6\text{i}$

**b** $\sqrt{-28} = \sqrt{28 \times (-1)} = \sqrt{28}\sqrt{-1}$
$= \sqrt{4}\sqrt{7}\sqrt{-1} = (2\sqrt{7})\text{i}$

You can use the rules of surds to manipulate imaginary numbers.

**Watch out** An alternative way of writing $(2\sqrt{7})\text{i}$ is $2\text{i}\sqrt{7}$. Avoid writing $2\sqrt{7}\text{i}$ as this can easily be confused with $2\sqrt{7\text{i}}$.

In a complex number, the real part and the imaginary part cannot be combined to form a single term.

- **Complex numbers can be added or subtracted by adding or subtracting their real parts and adding or subtracting their imaginary parts.**
- **You can multiply a real number by a complex number by multiplying out the brackets in the usual way.**

**Example 2**

Simplify each of the following, giving your answers in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$.

**a** $(2 + 5\text{i}) + (7 + 3\text{i})$ **b** $(2 - 5\text{i}) - (5 - 11\text{i})$ **c** $2(5 - 8\text{i})$ **d** $\frac{10 + 6\text{i}}{2}$

**a** $(2 + 5\text{i}) + (7 + 3\text{i}) = (2 + 7) + (5 + 3)\text{i}$
$= 9 + 8\text{i}$

Add the real parts and add the imaginary parts.

**b** $(2 - 5\text{i}) - (5 - 11\text{i}) = (2 - 5) + (-5 - (-11))\text{i}$
$= -3 + 6\text{i}$

Subtract the real parts and subtract the imaginary parts.

**c** $2(5 - 8\text{i}) = (2 \times 5) - (2 \times 8)\text{i} = 10 - 16\text{i}$

$2(5 - 8\text{i})$ can also be written as $(5 - 8\text{i}) + (5 - 8\text{i})$.

**d** $\frac{10 + 6\text{i}}{2} = \frac{10}{2} + \frac{6}{2}\text{i} = 5 + 3\text{i}$

First separate into real and imaginary parts.

**Example 4**

Solve the equation $z^2 + 6z + 25 = 0$.

Method 1 (Completing the square)

$z^2 + 6z = (z + 3)^2 - 9$

Because $(z + 3)^2 = (z + 3)(z + 3) = z^2 + 6z + 9$

$z^2 + 6z + 25 = (z + 3)^2 - 9 + 25 = (z + 3)^2 + 16$

$(z + 3)^2 + 16 = 0$

$(z + 3)^2 = -16$

$z + 3 = \pm\sqrt{-16} = \pm 4\text{i}$

$\sqrt{-16} = \sqrt{16 \times (-1)} = \sqrt{16}\sqrt{-1} = 4\text{i}$

$z = -3 \pm 4\text{i}$

$z = -3 + 4\text{i}, \quad z = -3 - 4\text{i}$

You can use your calculator to find the complex roots of a quadratic equation like this one.

Method 2 (Quadratic formula)

$$z = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 25}}{2}$$

Using $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-6 \pm \sqrt{-64}}{2}$$

$$z = \frac{-6 \pm 8\text{i}}{2} = -3 \pm 4\text{i}$$

$\sqrt{-64} = \sqrt{64 \times (-1)} = \sqrt{64}\sqrt{-1} = 8\text{i}$

$z = -3 + 4\text{i}, \quad z = -3 - 4\text{i}$

## Complex Numbers Practice Section A:

3 Simplify, giving your answers in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$.

**a** $2(7 + 2\text{i})$ **b** $3(8 - 4\text{i})$

**c** $2(3 + \text{i}) + 3(2 + \text{i})$ **d** $5(4 + 3\text{i}) - 4(-1 + 2\text{i})$

**e** $\dfrac{6 - 4\text{i}}{2}$ **f** $\dfrac{15 + 25\text{i}}{5}$

**g** $\dfrac{9 + 11\text{i}}{3}$ **h** $\dfrac{-8 + 3\text{i}}{4} - \dfrac{7 - 2\text{i}}{2}$

(P) 4 Write in the form $a + b\text{i}$, where $a$ and $b$ are simplified surds.

**a** $\dfrac{4 - 2\text{i}}{\sqrt{2}}$ **b** $\dfrac{2 - 6\text{i}}{1 + \sqrt{3}}$

5 Given that $z = 7 - 6\text{i}$ and $w = 7 + 6\text{i}$, find, in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$:

**a** $z - w$ **b** $w + z$

**Notation** Complex numbers are often represented by the letter $z$ or the letter $w$.

(E) 6 Given that $z_1 = a + 9\text{i}$, $z_2 = -3 + b\text{i}$ and $z_2 - z_1 = 7 + 2\text{i}$, find $a$ and $b$ where $a, b \in \mathbb{R}$. **(2 marks)**

(P) 7 Given that $z_1 = 4 + \text{i}$ and $z_2 = 7 - 3\text{i}$, find, in the form $a + b\text{i}$, where $a, b \in \mathbb{R}$:

**a** $z_1 - z_2$ **b** $4z_2$ **c** $2z_1 + 5z_2$

(P) 8 Given that $z = a + b\text{i}$ and $w = a - b\text{i}$, $a, b \in \mathbb{R}$, show that:

**a** $z + w$ is always real **b** $z - w$ is always imaginary

You can use complex numbers to find solutions to any quadratic equation with real coefficients.

- **If $b^2 - 4ac < 0$ then the quadratic equation $ax^2 + bx + c = 0$ has two distinct complex roots, neither of which are real.**

Solve the equation $z^2 + 6z + 25 = 0$.

**Method 1** (Completing the square)

$z^2 + 6z = (z + 3)^2 - 9$

$z^2 + 6z + 25 = (z + 3)^2 - 9 + 25 = (z + 3)^2 + 16$

$(z + 3)^2 + 16 = 0$

$(z + 3)^2 = -16$

$z + 3 = \pm\sqrt{-16} = \pm 4\text{i}$

$z = -3 \pm 4\text{i}$

$z = -3 + 4\text{i}, \quad z = -3 - 4\text{i}$

Because $(z + 3)^2 = (z + 3)(z + 3) = z^2 + 6z + 9$

$\sqrt{-16} = \sqrt{16 \times (-1)} = \sqrt{16}\sqrt{-1} = 4\text{i}$

You can use your calculator to find the complex roots of a quadratic equation like this one.

**Method 2** (Quadratic formula)

$$z = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 25}}{2}$$

$$= \frac{-6 \pm \sqrt{-64}}{2}$$

$$z = \frac{-6 \pm 8\text{i}}{2} = -3 \pm 4\text{i}$$

$z = -3 + 4\text{i}, \quad z = -3 - 4\text{i}$

Using $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\sqrt{-64} = \sqrt{64 \times (-1)} = \sqrt{64}\sqrt{-1} = 8\text{i}$

## Complex Numbers Practice Section B:

**Do not use your calculator in this exercise:**

**3** Solve each of the following equations. Write your answers in the form $a \pm b\text{i}$.

**a** $z^2 + 2z + 5 = 0$ **b** $z^2 - 2z + 10 = 0$ **c** $z^2 + 4z + 29 = 0$

**d** $z^2 + 10z + 26 = 0$ **e** $z^2 + 5z + 25 = 0$ **f** $z^2 + 3z + 5 = 0$

**4** Solve each of the following equations. Write your answers in the form $a \pm bi$.

**a** $2z^2 + 5z + 4 = 0$ **b** $7z^2 - 3z + 3 = 0$ **c** $5z^2 - z + 3 = 0$

**5** The solutions to the quadratic equation $z^2 - 8z + 21 = 0$ are $z_1$ and $z_2$. Find $z_1$ and $z_2$, giving each in the form $a \pm \text{i}\sqrt{b}$.

**6** The equation $z^2 + bz + 11 = 0$, where $b \in \mathbb{R}$, has distinct non-real complex roots. Find the range of possible values of $b$.

You can multiply complex numbers using the same technique that you use for multiplying brackets in algebra. You can use the fact that $i = \sqrt{-1}$ to simplify powers of i.

- $\mathbf{i^2 = -1}$

**Example 5**

Express each of the following in the form $a + bi$, where $a$ and $b$ are real numbers.

**a** $(2 + 3i)(4 + 5i)$ **b** $(7 - 4i)^2$

**a** $(2 + 3i)(4 + 5i) = 2(4 + 5i) + 3i(4 + 5i)$

Multiply the two brackets as you would with real numbers.

$= 8 + 10i + 12i + 15i^2$

$= 8 + 10i + 12i - 15$

Use the fact that $i^2 = -1$.

$= (8 - 15) + (10i + 12i)$

Add real parts and add imaginary parts.

$= -7 + 22i$

**b** $(7 - 4i)^2 = (7 - 4i)(7 - 4i)$

$= 7(7 - 4i) - 4i(7 - 4i)$

Multiply out the two brackets as you would with real numbers.

$= 49 - 28i - 28i + 16i^2$

$= 49 - 28i - 28i - 16$

Use the fact that $i^2 = -1$.

$= (49 - 16) + (-28i - 28i)$

Add real parts and add imaginary parts.

$= 33 - 56i$

## Complex Numbers Practice Section C:

(P) **2** **a** Simplify $(4 + 5i)(4 - 5i)$, giving your answer in the form $a + bi$.

**b** Simplify $(7 - 2i)(7 + 2i)$, giving your answer in the form $a + bi$.

**c** Comment on your answers to parts **a** and **b**.

**d** Prove that $(a + bi)(a - bi)$ is a real number for any real numbers $a$ and $b$.

(P) **3** Given that $(a + 3i)(1 + bi) = 25 - 39i$, find two possible pairs of values for $a$ and $b$.

**4** Write each of the following in its simplest form.

**a** $i^6$ **b** $(3i)^4$ **c** $i^5 + i$ **d** $(4i)^3 - 4i^3$

(P) **5** Express $(1 + i)^6$ in the form $a - bi$, where $a$ and $b$ are integers to be found.

(P) **6** Find the value of the real part of $(3 - 2i)^4$.

(P) **7** $f(z) = 2z^2 - z + 8$

Find: **a** $f(2i)$ **b** $f(3 - 6i)$

**Problem-solving**

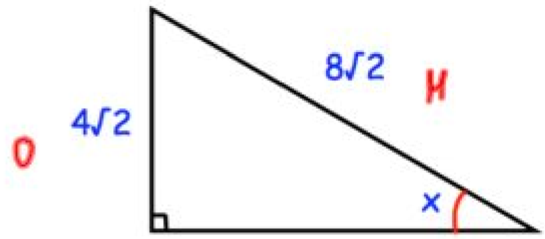
You can use the binomial theorem to expand $(a + b)^n$. ← Pure Year 1, Section 8.3

# Answers

## Trigonometry Practice Section A:

a)

Below is a right angled triangle.



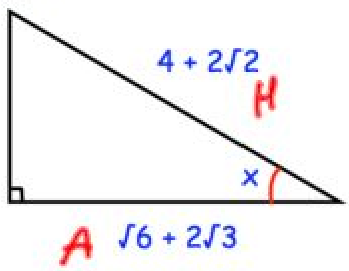

Show that angle x = 30°
Include all your working.

$$\sin x = \frac{O}{H} = \frac{4\sqrt{2}}{8\sqrt{2}} = \frac{1}{2}$$

$$\therefore x = 30^\circ$$

b)

Below is a right angled triangle.




Show that angle x = 30°
Include all your working.

$$\cos x = \frac{A}{H} = \frac{\sqrt{6}+2\sqrt{3}}{4+2\sqrt{2}} \times \frac{(4-2\sqrt{2})}{(4-2\sqrt{2})}$$

$$\frac{\cancel{4\sqrt{6}} - 2\sqrt{12} + 8\sqrt{3} - \cancel{4\sqrt{6}}}{16-8} = \frac{-2\sqrt{12}+8\sqrt{3}}{8}$$

$$= \frac{-4\sqrt{3}+8\sqrt{3}}{8} = \frac{4\sqrt{3}}{8}$$ **(2)**



$$\cos x = \frac{\sqrt{3}}{2} \quad \therefore x = 30^\circ$$

c)

Find the exact value of Sin(45°) + Cos(30°)

$\sin 45 = \frac{\sqrt{2}}{2}$

$\cos 30 = \frac{\sqrt{3}}{2}$

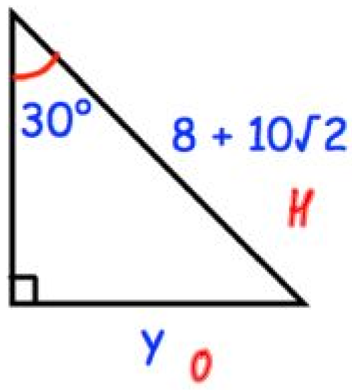
$\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}$

$\frac{\sqrt{2}+\sqrt{3}}{2}$

**(3)**

d)

Shown below is a right angled triangle.




Find the exact length of the side labelled y.

$$\sin 30 = \frac{y}{8+10\sqrt{2}}$$

$$\frac{1}{2} = \frac{y}{8+10\sqrt{2}}$$

$$4+5\sqrt{2} = y$$

$4+5\sqrt{2}$

**(4)**

## Algebraic Fractions Practice Section A:

1. a) $\frac{a^{\cancel{2}}}{\cancel{a}b}=\frac{a}{b}$ **A1**

b) $\frac{l}{m}\times\frac{m}{n}=\frac{l\cancel{m}}{\cancel{m}n}=\frac{l}{n}$ **A1**

c) $\frac{3x}{y}\div\frac{x}{z}=\frac{3x}{y}\times\frac{z}{x}=\frac{3\cancel{x}z}{y\cancel{x}}=\frac{3z}{y}$ **A1**

**Technique:** To divide by a fraction $\frac{a}{b}$, multiply by the reciprocal $\frac{b}{a}$

d) $\frac{6}{a}\div\frac{3}{a}=\frac{6}{a}\times\frac{a}{3}=\frac{\overset{2}{\cancel{6}}\cancel{a}}{\underset{1}{\cancel{3}}\cancel{a}}=\frac{2}{1}=2$ **A1**

e) $\frac{x(y-1)}{y(x+3)}\times\frac{y}{x}=\frac{\cancel{x}(y-1)\cancel{y}}{\cancel{y}(x+3)\cancel{x}}=\frac{y-1}{x+3}$ **A1**

f) $\frac{x}{y}\div\frac{x(x+1)}{y(y+2)}=\frac{x}{y}\times\frac{y(y+2)}{x(x+1)}$

$=\frac{\cancel{xy}(y+2)}{\cancel{yx}(x+1)}=\frac{y+2}{x+1}$ **A1**

g) $\frac{a-3}{5}\times\frac{10}{(a-3)(a+1)}=\frac{\overset{2}{\cancel{10}}\cancel{(a-3)}}{\underset{1}{\cancel{5}}\cancel{(a-3)}(a+1)}$ **M1**

$=\frac{2}{a+1}$ **A1**

h) $\frac{y-1}{4}\div\frac{(y-1)(y+3)}{8}=\frac{y-1}{4}\times\frac{8}{(y-1)(y+3)}$

$=\frac{\overset{2}{\cancel{8}}\cancel{(y-1)}}{\underset{1}{\cancel{4}}\cancel{(y-1)}(y+3)}$ **M1**

$=\frac{2}{y+3}$ **A1** **[10 Marks]**

2. a) $\frac{(5x+5)(x-2)}{(x+1)(2x-4)}=\frac{5\cancel{(x+1)}\cancel{(x-2)}}{\cancel{(x+1)}\times2\cancel{(x-2)}}$ **M1**

$=\frac{5}{2}$ **A1**

b) $\frac{(y^2+2y-3)(y-4)}{(y+3)(y^2-3y-4)}==\frac{(y-1)\cancel{(y+3)}\cancel{(y-4)}}{\cancel{(y+3)}\cancel{(y-4)}(y+1)}$ **M1**

**Tip:** Always try to factorise and check what can be cancelled before multiplying out the brackets

$=\frac{y-1}{y+1}$ **A1**

c) $\dfrac{a^2+6a+5}{a^2-a-6}\times\dfrac{a+2}{a+5}=\dfrac{(a+5)(a+1)}{(a-3)(a+2)}\times\dfrac{a+2}{a+5}$ **M1**

$=\dfrac{\cancel{(a+5)}(a+1)\cancel{(a+2)}}{(a-3)\cancel{(a+2)}\cancel{(a+5)}}=\dfrac{a+1}{a-3}$ **A1**

d) $\dfrac{x^2-2x-8}{x+2}\div\dfrac{x^2+x-20}{x-3}=\dfrac{x^2-2x-8}{x+2}\times\dfrac{x-3}{x^2+x-20}=\dfrac{(x-4)(x+2)}{x+2}\times\dfrac{x-3}{(x+5)(x-4)}$ **M1**

$=\dfrac{\cancel{(x-4)}\cancel{(x+2)}(x-3)}{\cancel{(x+2)}(x+5)\cancel{(x-4)}}$ **M1**

$=\dfrac{x-3}{x+5}$ **A1** **[9 Marks]**

3. a) $\dfrac{1}{4x}+\dfrac{1}{y}=\dfrac{1\times y}{4xy}+\dfrac{4x\times 1}{4xy}$ **M1**

$=\dfrac{y+4x}{4xy}$ **A1**

b) $\dfrac{2}{a}-\dfrac{3}{a+2}=\dfrac{2\times(a+2)}{a(a+2)}-\dfrac{3\times a}{a(a+2)}$ **M1**

$=\dfrac{2a+4-3a}{a(a+2)}$ **M1**

$=\dfrac{4-a}{a(a+2)}\left(\text{also allow }\dfrac{4-a}{a^2+2a}\right)$ **A1**

c) $\dfrac{5}{x-1}+\dfrac{3}{2x-2}=\dfrac{5}{x-1}+\dfrac{3}{2(x-1)}=\dfrac{2\times 5}{2(x-1)}+\dfrac{3}{2(x-1)}$ **M1**

> **Technique:** Save time by checking whether one denominator is a factor of another. Here, $(x-1)\times 2=2x-2$, so the first fraction can just be multiplied by 2 to make the denominators the same.

$=\dfrac{10+3}{2(x-1)}$ **M1**

$=\dfrac{13}{2(x-1)}\left(\text{also allow }\dfrac{13}{2x-2}\right)$ **A1** **[8 Marks]**

## Algebraic Fractions Practice Section B:

1. a) $\frac{ab}{ac} = \frac{\cancel{a}b}{\cancel{a}c} = \frac{b}{c}$ **A1**

b) $\frac{x}{y} \times \frac{y}{z} = \frac{x\cancel{y}}{\cancel{y}z} = \frac{x}{z}$ **A1**

c) $\frac{2m}{n} \div \frac{m}{p} = \frac{2m}{n} \times \frac{p}{m} = \frac{2\cancel{m}p}{n\cancel{m}} = \frac{2p}{n}$ **A1**

**Technique:** To divide by a fraction $\frac{a}{b}$, multiply by the reciprocal $\frac{b}{a}$

d) $\frac{4}{x} \div \frac{5}{x} = \frac{4}{x} \times \frac{x}{5} = \frac{4\cancel{x}}{5\cancel{x}} = \frac{4}{5}$ **A1**

e) $\frac{x(x+1)}{y(y+2)} \times \frac{y}{x} = \frac{\cancel{x}(x+1)\cancel{y}}{\cancel{y}(y+2)\cancel{x}} = \frac{x+1}{y+2}$ **A1**

f) $\frac{a}{b} \div \frac{a(a+4)}{b(b-2)} = \frac{a}{b} \times \frac{b(b-2)}{a(a+4)}$

$= \frac{\cancel{ab}(b-2)}{\cancel{ba}(a+4)} = \frac{b-2}{a+4}$ **A1**

g) $\frac{x+3}{2} \times \frac{4}{(x+4)(x+3)} = \frac{\overset{2}{\cancel{4}}\,\cancel{(x+3)}}{\underset{1}{\cancel{2}}(x+4)\cancel{(x+3)}}$ **M1**

$= \frac{2}{x+4}$ **A1**

h) $\frac{y+2}{3} \div \frac{(y+2)(y+5)}{9} = \frac{y+2}{3} \times \frac{9}{(y+2)(y+5)}$

$= \frac{\overset{3}{\cancel{9}}\,\cancel{(y+2)}}{\underset{1}{\cancel{3}}\cancel{(y+2)}(y+5)}$ **M1**

$= \frac{3}{y+5}$ **A1** **[10 Marks]**

2. a) $\frac{(3x-9)(x+1)}{(x-3)(2x+2)}=\frac{3\cancel{(x-3)}\cancel{(x+1)}}{\cancel{(x-3)}\times 2\cancel{(x+1)}}$ **M1**

$=\frac{3}{2}$ **A1**

b) $\frac{(y^2+4y+3)(y+4)}{(y+1)(y^2+6y+8)}=\frac{(y+3)\cancel{(y+1)}\cancel{(y+4)}}{\cancel{(y+1)}\cancel{(y+4)}(y+2)}$ **M1**

Tip: Always try to factorise and check what can be cancelled before multiplying out the brackets

$=\frac{y+3}{y+2}$ **A1**

c) $\frac{a^2+5a+6}{a^2+3a-4}\times\frac{a+4}{a+2}=\frac{(a+2)(a+3)}{(a-1)(a+4)}\times\frac{a+4}{a+2}$ **M1**

$=\frac{\cancel{(a+2)}(a+3)\cancel{(a+4)}}{(a-1)\cancel{(a+4)}\cancel{(a+2)}}=\frac{a+3}{a-1}$ **A1**

d) $\frac{x^2+7x+10}{x+5}\div\frac{x^2+3x+2}{x+3}=\frac{x^2+7x+10}{x+5}\times\frac{x+3}{x^2+3x+2}=\frac{(x+2)(x+5)}{x+5}\times\frac{x+3}{(x+1)(x+2)}$ **M1**

$=\frac{\cancel{(x+2)}\cancel{(x+5)}(x+3)}{\cancel{(x+5)}(x+1)\cancel{(x+2)}}$ **M1**

$=\frac{x+3}{x+1}$ **A1** **[9 Marks]**

3. a) $\frac{1}{x}+\frac{x}{2y}=\frac{1\times 2y}{2xy}+\frac{x\times x}{2xy}$ **M1**

$=\frac{2y+x^2}{2xy}$ **A1**

b) $\frac{3}{a}-\frac{2}{a+1}=\frac{3\times(a+1)}{a(a+1)}-\frac{2\times a}{a(a+1)}$ **M1**

$=\frac{3a+3-2a}{a(a+1)}$ **M1**

$=\frac{a+3}{a(a+1)}\left(\text{also allow } \frac{a+3}{a^2+a}\right)$ **A1**

c) $\frac{2}{x+3}+\frac{5}{2x+6}=\frac{2}{x+3}+\frac{5}{2(x+3)}=\frac{2\times 2}{2(x+3)}+\frac{5}{2(x+3)}$ **M1**

Technique: Save time by checking whether one denominator is a factor of another. Here, $(x + 3) \times 2 = 2x + 6$, so the first fraction can just be multiplied by 2 to make the denominators the same.

$=\frac{4+5}{2(x+3)}$ **M1**

$=\frac{9}{2(x+3)}\left(\text{also allow } \frac{9}{2x+6}\right)$ **A1** **[8 Marks]**

# Algebraic Fractions Practice Section C:

1. a) $\frac{3x^2}{xy^2} = \frac{3\cancel{x^2}^{x}}{\cancel{x}y^2} = \frac{3x}{y^2}$ **A1**

b) $\frac{x}{y(y-4)} \times \frac{y(y+2)}{x(x-1)} = \frac{\cancel{xy}(y+2)}{\cancel{yx}(y-4)(x-1)}$ **M1**

$= \frac{y+2}{(y-4)(x-1)}$ **A1**

c) $\frac{x(x-3)}{y(y-2)} \div \frac{x}{y(y+6)} = \frac{x(x-3)}{y(y-2)} \times \frac{y(y+6)}{x}$

**Technique:** To divide by a fraction $\frac{a}{b}$, multiply by the reciprocal $\frac{b}{a}$

$= \frac{\cancel{x}(x-3)\cancel{y}(y+6)}{\cancel{y}(y-2)\cancel{x}}$ **M1**

$= \frac{(x-3)(y+6)}{y-2}$ **A1** **[5 Marks]**

**Technique:** Save time by checking whether the denominators share a common factor. Here, both denominators have a common factor of $3x$, so the first fraction can just be multiplied by 2 to make the denominators the same

2. a) $\frac{2}{3x} + \frac{x+1}{6x} = \frac{2\times 2}{2\times 3x} + \frac{x+1}{6x}$ **M1**

$= \frac{4+x+1}{6x} = \frac{x+5}{6x}$ **A1**

b) $\frac{3}{a^2-2a-15} - \frac{1}{a^2-4a-5} = \frac{3}{(a-5)(a+3)} - \frac{1}{(a+1)(a-5)}$ **M1**

$= \frac{3\times(a+1)}{(a-5)(a+3)(a+1)} - \frac{1\times(a+3)}{(a+1)(a-5)(a+3)}$ **M1**

$= \frac{3a+3-a-3}{(a-5)(a+3)(a+1)}$ **M1**

$= \frac{2a}{(a-5)(a+3)(a+1)}$ **A1**

c) $\frac{5b-2}{b^2-4b-12} + \frac{4}{b+2} = \frac{5b-2}{(b+2)(b-6)} + \frac{4}{b+2}$ **M1**

**Technique:** Use the same technique from part a) to make the common denominator $(b+2)(b-6)$

$= \frac{5b-2}{(b+2)(b-6)} + \frac{4\times(b-6)}{(b+2)(b-6)}$ **M1**

$= \frac{5b-2+4b-24}{(b+2)(b-6)}$ **M1**

$= \frac{9b-26}{(b+2)(b-6)}$ **A1** **[10 Marks]**

**Technique:** First make sure the

# Algebraic Fractions Practice Section D:

1. a) $\frac{2x}{x^{2}y}=\frac{2\cancel{x}}{\cancel{x^{2}}y}=\frac{2}{xy}$ **A1**

b) $\frac{x}{y(y+6)}\times\frac{y(y+3)}{x(x-2)}=\frac{\cancel{xy}(y+3)}{\cancel{yx}(y+6)(x-2)}$ **M1**

$=\frac{y+3}{(y+6)(x-2)}$ **A1**

c) $\frac{x(x+3)}{y(y+1)}\div\frac{x}{y(y-2)}=\frac{x(x+3)}{y(y+1)}\times\frac{y(y-2)}{x}$

**Technique:** To divide by a fraction $\frac{a}{b}$, multiply by the reciprocal $\frac{b}{a}$

$=\frac{\cancel{x}(x+3)\cancel{y}(y-2)}{\cancel{y}(y+1)\cancel{x}}$ **M1**

$=\frac{(x+3)(y-2)}{y+1}$ **A1** **[5 Marks]**

**Technique:** Save time by checking

$y+1$

2. a) $\frac{3}{4x}+\frac{x+1}{2x}=\frac{3}{4x}+\frac{2\times(x+1)}{2\times 2x}$ **M1**

**Technique:** Save time by checking whether the denominators share a common factor. Here, both denominators have a common factor of $2x$, so the second fraction can just be multiplied by 2 to make the denominators the same.

$=\frac{3+2x+2}{4x}=\frac{2x+5}{4x}$ **A1**

b) $\frac{2}{a^{2}+2a-3}-\frac{1}{a^{2}+5a+6}=\frac{2}{(a+3)(a-1)}-\frac{1}{(a+2)(a+3)}$ **M1**

$=\frac{2\times(a+2)}{(a+3)(a-1)(a+2)}-\frac{1\times(a-1)}{(a+2)(a+3)(a-1)}$ **M1**

$=\frac{2a+4-a+1}{(a+3)(a-1)(a+2)}$ **M1**

$=\frac{a+5}{(a+3)(a-1)(a+2)}$ **A1**

c) $\frac{3b+4}{b^{2}+b-12}+\frac{2}{b+4}=\frac{3b+4}{(b+4)(b-3)}+\frac{2}{b+4}$ **M1**

$=\frac{3b+4}{(b+4)(b-3)}+\frac{2\times(b-3)}{(b+4)(b-3)}$ **M1**

**Technique:** Use the same technique from part a) to make the common denominator $(b+4)(b-3)$

$=\frac{3b+4+2b-6}{(b+4)(b-3)}$ **M1**

$=\frac{5b-2}{(b+4)(b-3)}$ **A1** **[10 Marks]**

3. $\frac{3x+2}{x^{2}-4}=\frac{3x+2}{(x+2)(x-2)}$ **M1**

**Technique:** First make sure the denominator only contains linear expressions. To write a fraction $\frac{f(x)}{g(x)h(x)}$ (where $g(x)$ and $h(x)$ are linear expressions) as partial fractions, equate it with the expression $\frac{A}{g(x)}+\frac{B}{h(x)}$ and find the constants $A$ and $B$.

Let $\frac{3x+2}{(x+2)(x-2)}\equiv\frac{A}{x+2}+\frac{B}{x-2}\equiv\frac{A(x-2)+B(x+2)}{(x+2)(x-2)}$

$\therefore 3x+2\equiv A(x-2)+B(x+2)$ **M1**

Substitute $x=2$:

$3(2)+2=A(2-2)+B(2+2)$

$\therefore 8=4B$

$\therefore B=2$ **M1**

Substitute $x=-2$:

$3(-2)+2=A(-2-2)+B(-2+2)$

$\therefore -4=-4A$

$\therefore A=1$ **M1**

$\therefore \frac{3x+2}{(x+2)(x-2)}\equiv\frac{1}{x+2}+\frac{2}{x-2}$ **A1** **[5 Marks]**

**Alternatively:** Instead of using substitution, you could also equate coefficients to find the values of $A$ and $B$.

## Proofs Practice Section A

1. $(2n+1)^2 - (2n+1)$

$= 4n^2 + 4n + 1 - 2n - 1$ expand the brackets

$= 4n^2 + 2n$ collect the terms

$= 2(2n^2 + n)$ Take out a factor of two

Since it is a multiple of two, it is an even number

2. $\dfrac{6c^3 + 30c}{3c^2 + 15} = \dfrac{3(2c^3 + 10c)}{3(c^2 + 5)}$ factorise

$= \dfrac{2c^3 + 10c}{c^2 + 5}$ divide/cancle

$= \dfrac{2c(c^2 + 5)}{c^2 + 5}$ factorise

$= 2c$ divide/cancle

Since it is a multiple of two, it is an even number

3. $n + (n+1) + (n+2) + (n+3)$

$= 4n + 6$ collect the terms

$= 2(2n+3)$ take out a factor of 2

Since it is a multiple of two, it is an even number

4. The $n^{th}$ term is $6n+1$

$$(6(n+1)+1)^2 - (6n+1)^2$$
$$= (6n+6+1)^2 - (6n+1)^2$$
$$= (6n+7)^2 - (6n+1)^2$$
$$= 36n^2 + 84n + 49 - 36n^2 - 12n - 1$$
$$= 72n + 48$$
$$= 24(3n+2)$$

Since it is 2 numbers from the sequence, it is a mutiple of 24 for the disference of two squares

5. $(n(n+1)) + (n+1)$
$$= n^2 + n + n + 1$$
$$= n^2 + 2n + 1$$
$$= (n+1)^2$$
The result is a square number

6. $(n+1)^2 - n^2 = n + (n+1)$
$$n^2 + 2n + 1 - n^2 = 2n + 1$$
$$2n + 1 = 2n + 1$$

## Proofs Practice Section B

**1)** C

**2)** C

**3)** B

**4)** D

**5)** $n+1+n+2=2n+3$
So, as $2n$ is divisible by $2$ and $3$ is an odd number. Therefore, the sum of two consecutive whole numbers is always an odd number.

**6)** $(25n^2+40n+16)-(25n^2-40n+16)$
$=80n$
$=4(20n)$
So, as $80$ is divisible by $4$ then $(5n+4)^2-(5n-4)^2$ is a multiple of 4, for all positive integer values of n.

**7)** $(25n^2+20n+4)-(25n^2-20n+4)$
$=40n$
$=8(5n)$
So, as $40$ is divisible by $8$ then $(5n+2)^2-(5n-2)^2$ is a multiple of 8, for all positive integer values of n.

**8)** Sum of two consecutive integers:-
$n+n-1$
$=2n-1$
Difference between the squares of two consecutive integers:-
$(n)^2-(n-1)^2$
$=(n^2)-(n^2-2n+1)$
$=2n-1$
So they are equal

## Matrices Practice Section A

**1.** **(a)** $2\times 2$
**(b)** $2\times 1$
**(c)** $2\times 3$
**(d)** $1\times 3$
**(e)** $1\times 2$
**(f)** $3\times 3$

**2.** **(a)** $\begin{pmatrix} 8 & -1 \\ 1 & 4 \end{pmatrix}$
**(b)** $\begin{pmatrix} 2 & 2 \\ -2 & 5 \end{pmatrix}$
**(c)** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

**3.** **(a)** Not possible
**(b)** $\begin{pmatrix} -2 \\ 3 \end{pmatrix}$
**(c)** $\begin{pmatrix} 1 & 1 & 4 \end{pmatrix}$
**(d)** Not possible
**(e)** $\begin{pmatrix} 3 & -1 & 4 \end{pmatrix}$
**(f)** Not possible
**(g)** $\begin{pmatrix} -3 & 1 & -4 \end{pmatrix}$

**4.** $a=6, b=3, c=2, d=-1$

**5.** $a=4, b=3, c=5$

**6.** $a=2, b=-2, c=2, d=1, e=-1, f=3$

## Matrices Practice Section B

1. (a) $\begin{pmatrix} 6 & 0 \\ 12 & -18 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 0 \\ 2 & -3 \end{pmatrix}$

   (c) $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$

2. $k = 3, x = -1$
3. $a = 3, b = -3.5, c = -1, d = 2$
4. $a = 5, b = 5, c = -2, d = 2$
5. $k = \dfrac{3}{2}$

## Matrices Practice Section C

1. (a) $1 \times 2$

   (b) $3 \times 3$

   (c) $1 \times 2$

   (d) $2 \times 2$

   (e) $2 \times 3$

   (f) $3 \times 2$

2. (a) $\begin{pmatrix} 3 \\ 5 \end{pmatrix}$

   (b) $\begin{pmatrix} -2 & 1 \\ -4 & 7 \end{pmatrix}$

3. (a) $\begin{pmatrix} -3 & -2 & -1 \\ 3 & 3 & 0 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & -4 \\ 0 & 9 \end{pmatrix}$

4. (a) Not possible

   (b) $\begin{pmatrix} -6 & -4 \\ -3 & -2 \end{pmatrix}$

   (c) Not possible

   (d) $\begin{pmatrix} 7 \\ 0 \end{pmatrix}$

   (e) $(-8)$

   (f) $\begin{pmatrix} -7 & -7 \end{pmatrix}$

5. $\begin{pmatrix} 2 & 6-a & 2a \\ 1 & 4 & -2 \end{pmatrix}$
6. $\begin{pmatrix} 3x+2 & 0 \\ 0 & 3x+2 \end{pmatrix}$

## Matrices Practice Section D

**1.** **(a)** Non-singular. Inverse = $\begin{pmatrix} 1 & 0.5 \\ 2 & 1.5 \end{pmatrix}$

**(b)** Singular

**(c)** Singular

**(d)** Non-singular. Inverse = $\begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$

**(e)** Singular

**(f)** Non-singular. Inverse = $\begin{pmatrix} -0.2 & 0.3 \\ 0.6 & -0.4 \end{pmatrix}$

**2.** **(a)** $-3$

**(b)** $-5$

**(c)** $\frac{1}{4}$

**6.** **(a)** $\mathbf{A} = \mathbf{C}^{-1}$

**(b)** $\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$

**7.** $\begin{pmatrix} 2 & 4 & -3 \\ 0 & 1 & 2 \end{pmatrix}$

**8.** $\begin{pmatrix} 1 & 3 \\ -2 & 1 \\ 0 & -1 \end{pmatrix}$

**9.** **(a)** $\frac{1}{2ab}\begin{pmatrix} 2b & -b \\ -4a & 3a \end{pmatrix}$

**(b)** $\begin{pmatrix} -3 & 2 \\ -1 & 3/2 \end{pmatrix}$

## Complex Numbers Practice Section A

**3 a** $14+4\mathrm{i}$

**b** $24-12\mathrm{i}$

**c** $(6+2\mathrm{i})+(6+3\mathrm{i})=(6+6)+\mathrm{i}(2+3)$
$=12+5\mathrm{i}$

**d** $(20+15\mathrm{i})+(4-8\mathrm{i})=(20+4)+\mathrm{i}(15-8)$
$=24+7\mathrm{i}$

**e** $\frac{6-4\mathrm{i}}{2}=\frac{6}{2}-\frac{4}{2}\mathrm{i}$
$=3-2\mathrm{i}$

**f** $\frac{15+25\mathrm{i}}{5}=\frac{15}{5}-\frac{25}{5}\mathrm{i}$
$=3+5\mathrm{i}$

**g** $\frac{9+11\mathrm{i}}{3}=\frac{9}{3}+\frac{11}{3}\mathrm{i}$
$=3+\frac{11}{3}\mathrm{i}$

**h** $\frac{-8+3\mathrm{i}}{4}-\frac{7-2\mathrm{i}}{2}=-2+\frac{3}{4}\mathrm{i}-\frac{7}{2}+\mathrm{i}$
$=\left(-\frac{4}{2}-\frac{7}{2}\right)+\mathrm{i}\left(\frac{3}{4}+\frac{4}{4}\right)$
$=-\frac{11}{2}+\frac{7}{4}\mathrm{i}$

**4 a** $\frac{4-2\mathrm{i}}{\sqrt{2}}=\frac{4-2\mathrm{i}}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$
$=\frac{4\sqrt{2}-2\sqrt{2}\mathrm{i}}{2}$
$=2\sqrt{2}-\mathrm{i}\sqrt{2}$

**b** $\frac{2-6\mathrm{i}}{1+\sqrt{3}}=\frac{2-6\mathrm{i}}{1+\sqrt{3}}\times\frac{(1-\sqrt{3})}{(1-\sqrt{3})}$
$=\frac{2-2\sqrt{3}-6\mathrm{i}+6\sqrt{3}\mathrm{i}}{1-3}$
$=\frac{2-2\sqrt{3}}{-2}+\frac{6\sqrt{3}-6}{-2}\mathrm{i}$
$=\left(-1+\sqrt{3}\right)+\left(3-3\sqrt{3}\right)\mathrm{i}$

**5 a** $z-w=(7-6\mathrm{i})-(7+6\mathrm{i})$
$=(7-7)+\mathrm{i}(-6-6)$
$=-12\mathrm{i}$

**b** $w+z=(7+6\mathrm{i})+(7-6\mathrm{i})$
$=(7+7)+\mathrm{i}(6-6)$
$=14$

**6** $7+2\mathrm{i}=z_2-z_1$
$=(-3+b\mathrm{i})-(a+9\mathrm{i})$
$=(-3-a)+(b-9)\mathrm{i}$

Equate real parts:
$7=-3-a\Rightarrow a=-10$

Equate imaginary parts:
$2=b-9\Rightarrow b=11$

**7 a** $z_1 - z_2 = (4+i)-(7-3i)$
$= (4-7)+i(1--3)$
$= -3+4i$

**b** $4z_2 = 4(7-3i)$
$= 28-12i$

**c** $2z_1 + 5z_2 = 2(4+i)+5(7-3i)$
$= 8+2i+35-15i$
$= 43-13i$

**8 a** $z+w = (a+bi)+a-bi$
$= 2a$
So $z+w$ is always real.

**b** $z-w = (a+bi)-(a-bi)$
$= 2bi$
So $z-w$ is always imaginary.

## Complex Numbers Practice Section B

**3** [Note that **3a**, **3b** & **3c** use the quadratic formula, and **3d**, **3e** & **3f** use completion of the square]

**a** $a=1,\ b=2,\ c=5$

$$z=\frac{-2\pm\sqrt{(4-20)}}{2}$$
$$=\frac{-2\pm 4\mathrm{i}}{2}$$
$$=-1\pm 2\mathrm{i}$$

**b** $a=1,\ b=-2,\ c=10$

$$z=\frac{2\pm\sqrt{(4-40)}}{2}$$
$$=\frac{2\pm 6\mathrm{i}}{2}$$
$$=1\pm 3\mathrm{i}$$

**c** $a=1,\ b=4,\ c=29$

$$z=\frac{-4\pm\sqrt{(16-116)}}{2}$$
$$=\frac{-4\pm 10\mathrm{i}}{2}$$
$$=-2\pm 5\mathrm{i}$$

**d**
$$z^2+10z+26=0$$
$$(z+5)^2-25+26=0$$
$$(z+5)^2=-1$$
$$z+5=\pm\mathrm{i}$$
$$z=-5\pm\mathrm{i}$$

**e**
$$z^2+5z+25=0$$
$$\left(z+\frac{5}{2}\right)^2-\frac{25}{4}+25=0$$
$$\left(z+\frac{5}{2}\right)^2=-\frac{75}{4}$$
$$z+\frac{5}{2}=\pm\frac{\mathrm{i}\sqrt{25\times 3}}{\sqrt{4}}$$
$$z=-\frac{5}{2}\pm\frac{5\mathrm{i}\sqrt{3}}{2}$$

**f**
$$z^2+3z+5=0$$
$$\left(z+\frac{3}{2}\right)^2-\frac{9}{4}+5=0$$
$$\left(z+\frac{3}{2}\right)^2=-\frac{11}{4}$$
$$z+\frac{3}{2}=\pm\frac{\mathrm{i}\sqrt{11}}{2}$$
$$z=-\frac{3}{2}\pm\frac{\mathrm{i}\sqrt{11}}{2}$$

**4** [Note that **4a** uses completion of the square and **4b** & **4c** use the quadratic formula]

**a**

$$2z^2+5z+4=0$$
$$z^2+\frac{5}{2}z+2=0$$
$$\left(z+\frac{5}{4}\right)^2-\frac{25}{16}+2=0$$
$$\left(z+\frac{5}{4}\right)^2=-\frac{7}{16}$$
$$z+\frac{5}{4}=\pm\frac{\mathrm{i}\sqrt{7}}{4}$$
$$z=-\frac{5}{4}\pm\frac{\mathrm{i}\sqrt{7}}{4}$$

**b** Use the quadratic formula with $a=7$, $b=-3$ and $c=3$

$$z=\frac{-(-3)\pm\sqrt{(-3)^2-(4)(7)(3)}}{2(7)}$$
$$=\frac{3\pm\sqrt{9-84}}{14}$$
$$=\frac{3\pm\sqrt{-75}}{14}$$
$$=\frac{3}{14}\pm\frac{5\sqrt{3}}{14}\mathrm{i}$$

**c** Use the quadratic formula with $a=5$, $b=-1$ and $c=3$

$$z=\frac{-(-1)\pm\sqrt{(-1)^2-4(5)(3)}}{2(5)}$$
$$=\frac{1\pm\sqrt{1-60}}{10}$$
$$=\frac{1}{10}\pm\frac{\sqrt{59}}{10}\mathrm{i}$$

**5** Method 1: Completing the square

$$(z-4)^2-16+21=0$$
$$(z-4)^2=-5$$
$$z-4=\pm\mathrm{i}\sqrt{5}$$
$$z=4\pm\mathrm{i}\sqrt{5}$$

So $z_1=4+\mathrm{i}\sqrt{5}$ and $z_2=4-\mathrm{i}\sqrt{5}$

Method 2: using the quadratic formula

$$z=\frac{-(-8)\pm\sqrt{(-8)^2-4(1)(21)}}{2(1)}$$
$$=\frac{8\pm\sqrt{-20}}{2}$$
$$=\frac{8\pm\mathrm{i}\sqrt{4}\sqrt{5}}{2}$$
$$=4\pm\mathrm{i}\sqrt{5}$$

**6** $z^2+bz+11=0$ has two distinct complex roots when

$$b^2-4(1)(11)<0$$
$$b^2<44$$
$$-\sqrt{44}<b<\sqrt{44}$$
or $-2\sqrt{11}<b<2\sqrt{11}$

# Complex Numbers Practice Section C

**2** **a** $(4+5\text{i})(4-5\text{i}) = 16-20\text{i}+20\text{i}-25\text{i}^2$
$= 16-20\text{i}+20\text{i}+25$
$= 41$

**b** $(7-2\text{i})(7+2\text{i}) = 49+14\text{i}-14\text{i}-4\text{i}^2$
$= 49+14\text{i}-14\text{i}+4$
$= 53$

**c** The answers to **2a** and **2b** are both real.

**d** Let $a$ and $b$ be any real numbers.
$(a+b\text{i})(a-b\text{i}) = a^2 - ab\text{i} + ab\text{i} - b^2\text{i}^2$
$= a^2 - ab\text{i} + ab\text{i} + b^2$
$= a^2 + b^2$

For any $a$ and $b$, the imaginary parts cancel (i.e. they sum to zero), so the answer is always real.

**3** $$(a+3\mathrm{i})(1+b\mathrm{i})=25-39i$$

$$a+ab\mathrm{i}+3\mathrm{i}+3b\mathrm{i}^2=25-39\mathrm{i}$$

$$a+ab\mathrm{i}+3\mathrm{i}-3b=25-39\mathrm{i}$$

$$(a-3b)+(ab+3)\mathrm{i}=25-39\mathrm{i}$$

Equating real parts:

$$a-3b=25$$

$$a=3b+25 \qquad \textbf{(1)}$$

Equating imaginary parts:

$$ab+3=-39 \qquad \textbf{(2)}$$

Substituting (**1**) into (**2**):

$$(3b+25)b+3=-39$$

$$3b^2+25b+3=-39$$

$$3b^2+25b+42=0$$

$$(3b+7)(b+6)=0$$

So $b=\frac{-7}{3}$ or $b=-6$

Substituting $b=-6$ into (**1**):

$$a(-6)+3=-39$$

$$a=7$$

Substituting $b=\frac{-7}{3}$ into (**1**):

$$a\left(-\frac{7}{3}\right)+3=-39$$

$$a=18$$

Hence, the two pairs of values are:

$a=7,\ b=-6$

$a=18,\ b=-\frac{7}{3}$

**4** **a** $i\times i\times i\times i\times i\times i=i^2\times i^2\times i^2$

$$=-1\times-1\times-1$$
$$=-1$$

**b** $3i\times3i\times3i\times3i=81(i\times i\times i\times i)$

$$=81(i^2\times i^2)$$
$$=81(-1\times-1)$$
$$=81$$

**c** $(i\times i\times i\times i\times i)+i=(i^2\times i^2\times i)+i$

$$=(-1\times-1\times i)+i$$
$$=i+i$$
$$=2i$$

**d** $(4i)^3-4i^3=(4i\times4i\times4i)-4(i\times i\times i)$

$$=64(i\times i\times i)-4(i\times i\times i)$$
$$=60(i\times i\times i)$$
$$=60(-1\times i)$$
$$=-60i$$

**5** To expand $(1+i)^6$, use the binomial expansion of $(a+b)^6$

$$(a+b)^6=a^6+6a^5b+15a^4b^2+20a^3b^2$$
$$+15a^2b^4+6ab^5+b^6$$

Substitute $a=1$ and $b=i$:

$$(1+i)^6=(1)^6+6(1)^5(i)+15(1)^4(i)^2+20(1)^3(i)^3$$
$$+15(1)^2(i)^4+6(1)(i)^5+(i)^6$$

$$(1+i)^6=1+6i+15i^2+20i^3+15i^4+6i^5+i^6$$

[Use $i^2=-1, i^3=-i, i^4=1, i^5=i$ and $i^6=-1$]

$$(1+i)^6=1+6i-15-20i+15+6i-1$$
$$=-8i$$

So $a=0$ and $b=-8$

**6** To expand $(3-2\mathrm{i})^4$, use the binomial expansion of $(a+b)^4$:

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

Substitute $a = 3$ and $b = 2\mathrm{i}$:

$$(3-2\mathrm{i})^4 = (3)^4 + 4(3)^3(-2\mathrm{i}) + 6(3)^2(-2\mathrm{i})^2 + 4(3)(-2\mathrm{i})^3 + (-2\mathrm{i})^4$$

$$(3-2\mathrm{i})^4 = 81 - 216\mathrm{i} + 216\mathrm{i}^2 - 96\mathrm{i}^3 + 16\mathrm{i}^4$$

[Use $\mathrm{i}^2 = -1, \mathrm{i}^3 = -\mathrm{i}$ and $\mathrm{i}^4 = 1$]

$$\begin{aligned}(3-2\mathrm{i})^4 &= 81 - 216\mathrm{i} - 216 + 96\mathrm{i} + 16 \\ &= -119 - 120\mathrm{i}\end{aligned}$$

So the real part of $(3-2\mathrm{i})^4$ is $-119$

**7** **a** 
$$\begin{aligned}\mathrm{f}(2\mathrm{i}) &= 2(2\mathrm{i})^2 - (2\mathrm{i}) + 8 \\ &= 2(4\mathrm{i}^2) - 2\mathrm{i} + 8 \\ &= 8(-1) - 2\mathrm{i} + 8 \\ &= -2\mathrm{i}\end{aligned}$$

**b** $\mathrm{f}(3-6\mathrm{i})$

$$\begin{aligned}&= 2(3-6\mathrm{i})^2 - (3-6\mathrm{i}) + 8 \\ &= 2(9 - 36\mathrm{i} + 36\mathrm{i}^2) - 3 + 6\mathrm{i} + 8 \\ &= 18 - 72\mathrm{i} + 72\mathrm{i}^2 - 3 + 6\mathrm{i} + 8 \\ &= 18 - 72\mathrm{i} - 72 - 3 + 6\mathrm{i} + 8 \\ &= -49 - 66\mathrm{i}\end{aligned}$$